

# Sequential Decoding With a Noisy Carrier Reference

J. W. Layland

Communications Systems Research Section

*An approximate analysis of the effect of a noisy carrier reference on the performance of sequential decoding is presented. The limitations of the analysis are discussed and steps are described that could be taken to extend the performance region over which the model used produces accurate, rather than merely bounding, results.*

## I. Introduction

Convolutional encoding with sequential decoding is a very powerful technique for communicating at low error probability with deep space probes. It has been used successfully with Pioneer 9 and 10, and is planned for use on Helios. Most, if not all, of the performance data for this coding technique have been developed without regard to the effects of noisy reference signals in carrier or subcarrier tracking loops. These effects must be known with fair accuracy for the optimal design of telemetry links with sequential decoding.

## II. Sequential Decoding—The Computation Problem

The convolutional codes which are sequentially decoded typically have a large enough constraint length so that the undetected error probability out of the decoder is negligible compared to the probability that a block cannot be successfully decoded in the time allowed. Thus, the limiting factor for sequential decoding is the proba-

bility that large amounts of computation are required to decode a frame of the code, rather than the probability of error. Experimental and theoretical work has shown that the distribution of the number of computations  $c_1$  needed by the decoder to penetrate 1 bit deeper into the convolutional code tree has a Pareto distribution

$$\Pr\{c_1 > x\} \sim kx^{-\alpha} \quad (1)$$

The exponent  $\alpha$  is the noisy channel error exponent (Ref. 1), and  $k$  is a small constant, found by Heller (Ref. 2) to be 1.9.

The computation distribution is somewhat changed when an entire code frame is considered. The number of computations needed by the decoder to penetrate from a depth of  $N - 1$  to a depth of  $N$  is certainly not independent of the number of computations needed to penetrate from depth  $N$  to depth  $N + 1$ . However, the number of computations needed to penetrate from depth  $N - 1$  to depth  $N$  is independent of the number of computations needed to penetrate from depth  $N + j - 1$  to depth  $N + j$ ,

if  $|j|$  is large enough. The magnitude  $|j|$  which is large enough to establish independence is believed to be a function of the signal-to-noise ratio (SNR). The Pareto distribution has the property that for moderately large  $N$ , the probability of a single long computation of length  $2N$  is much greater than the probability of two smaller computations, each of length approximately  $N$ . As a result, whenever the number of computations needed to decode a code frame is large, its distribution is dominated by single long computations, representing decoder penetration from  $M - l$  to  $M$ , for some  $M, l$ . Where the number of computations is small, however, the distribution function represents the sum of many small computations. This fact is important and will be used later.

Experimental distributions of the number of computations needed to decode the Pioneer 10 frame of 192-bit length were developed at NASA Ames Research Center (Ref. 3) and are reproduced here for convenience in Fig. 1. A curve of decoder erasure probability vs. SNR can be derived from this figure by projecting the curves on the plane defined by a fixed number of computations per frame.

### III. Carrier Loop Effects

The receivers of the DSN tracking stations use a narrowband phase-locked loop, tracking the carrier component of the signal received from the spacecraft, to provide a coherent reference for demodulation of the telemetry sidebands on that signal. The bandwidth of the phase-locked loop is generally wide enough to track out received doppler, yet narrow with respect to the telemetry data rate, so that the phase of the reference signal is essentially constant while several tens of bits are being received. If, for reasons of received noise or otherwise, a phase error  $\phi$  exists between the received carrier and the local carrier reference, the amplitude of the signal entering the decoder is degraded by a factor  $\cos \phi$  while that phase error  $\phi$  persists.

The probability distribution of the phase error  $\phi$  in a phase-locked loop has been derived elsewhere (Ref. 4) to be

$$P(\phi) = \frac{\exp(\rho_L \cos \phi)}{2\pi I_0(\rho_L)} \quad (2)$$

where  $I_0(-)$  is the zeroth-order modified Bessel function and  $\rho_L$  is  $2P_c/N_0W_L$ .

Lindsey (Ref. 5) has used this phase error distribution to derive performance curves for the biorthogonal block

code which account for noise in the reference signal under the (reasonable) assumption that the phase error  $\phi$  is constant while a code block is being received. The validity of this assumption depends merely upon the bandwidth of the phase-locked loop being narrow with respect to the rate at which code blocks are received. The theoretical bit-error probability curves, which are functions of bit SNR, can thus be considered functions of the phase error  $\phi$  that existed while each block was being received, and of the bit SNR that would exist if the carrier reference were perfect. Averaging over the probability distribution of phase error  $\phi$  results in performance curves which show the expected bit-error probability of the coded system, and account correctly for the losses due to a noisy carrier reference.

Heller and Jacobs (Ref. 6) have recently used the same technique to estimate the performance of optimally decoded, short-constraint-length convolutional codes with a noisy carrier reference. They argue that the averaging over  $\phi$  is valid whenever the phase error is relatively constant over a period of time which is long with respect to the constraint length of the code.

For block coding, the situation in which the phase error  $\phi$  varies during a code frame has been approximately analyzed by Tausworthe (Ref. 7), who developed a formula for interpolation between performance with constant phase (Ref. 5) and the performance expected with very rapidly varying phase error. The interpolation parameter is a function of the time-bandwidth product for the code-block integration time and the phase-locked-loop bandwidth. This same interpolation formula should be equally valid for short-constraint-length convolutional codes if we could reliably define the "integration time" of these codes.

For sequential decoding, if the phase error  $\phi$  is essentially constant over a frame of data, then it is clear that we can average the erasure probability curves conditioned on bit SNR (and  $\phi$ ) over the distribution of phase error, and derive a valid estimate of decoding performance with a noisy reference. This condition, however, requires that the phase-locked loop be extremely narrow with respect to data rate, an unrealistic assumption which, furthermore, does not appear to be necessary.

Let us consider the characteristics of the distribution of the number of computations per frame in the region where the number of computations is large. As noted before, the computations on any block in this region are dominated by single large computations that result from

the decoder extending its penetration from depth  $M - l$  to depth  $M$ , for some  $M$  and some  $l$  much less than the frame length. If the phase error  $\phi$  is essentially constant for these  $l$  or more bits, then the distribution of computations can be considered as being conditioned on  $\phi$  for large numbers of computations per frame.

If the total number of computations in a frame is small, then that computation must almost certainly have resulted as the sum of a (possibly large) number of independent searches, each with its own value of  $\phi$ . The degradation in the computation distribution which results from phase jitter in this case is less than the degradation which would result if the phase error were constant over the entire frame. A lower bound on the degradation can be determined by assuming that the phase error is independent from bit to bit and computing the expected loss in signal amplitude into the decoder by averaging over the distribution of  $\phi$ .

The pseudo-theoretical distribution of decoding computations which results from treating phase error as constant over an entire frame and averaging the perfect-reference computation distribution over the phase error distribution has thus two levels of validity: It is an accurate estimate of the low-probability, long computation events, which correspond to erasures in a typical system; and it represents an upper bound to the degradation due to phase error for the sums of several short searches, which occur with higher probability and which seldom represent erasures.

To obtain numerical results, the probability distribution family shown in Fig. 1 was approximated by functions of bit SNR ( $R$ ), and average number of computations per bit ( $N$ ). The chosen approximating functions are of the form

$$\Pr\{c_L > N*L\} = \exp \left\{ \sum_{\substack{n=-1,1 \\ r=0,2}} A_{n,r} R^r (\ln N)^n \right\} \quad (3)$$

The coefficients  $\{A_{n,r}\}$  were determined by a two-dimensional, least-squares polynomial fit, and appear in Table 1. The frame length is  $L$ .

The solid lines of Fig. 1 show this approximation. Having thus been defined as functions of bit SNR, it is a trivial task to express these distributions as functions of total bit SNR and carrier phase error  $\phi$ , and to numerically integrate them over the distribution of  $\phi$  (Eq. 2) for various values of the carrier tracking loop SNR. Figure 2 shows the computed distribution of the

computation for a fixed input symbol-error probability of 6%. The loop SNRs and bit SNRs correspond to the Pioneer 10 modulation index and its range of data rates. This same family of curves has been generated experimentally by C. Grauling and J. Wilcher (Ref. 8), and is reproduced here as Fig. 3. Notice that, in general, the computed distributions predict a considerably poorer performance than observed experimentally.

To understand why the experimental and computed results disagree, we need observe that decoding performance with noisy reference depends to some significant extent upon the decoding performance at very low SNR. For large values of  $\phi$ , the effective SNR at decoder input is very low, and is outside of the region enclosed by the data of Fig. 1. We are thus depending upon the valid extrapolation of Eq. (3) of Table 1. This extrapolation does not follow the trends observed in some recently developed experimental distributions (Ref. 9, Fig. 2): both the exponent  $\alpha$  and the probability-axis intercept remain too high at extremely low SNR.

We can induce a better fit at low SNR by including low-SNR hypothetical data in the data set, as shown in Fig. 4. The exponent of this curve corresponds theoretically to  $E_b/N_0 = -2.0$  dB (Ref. 10). At extremely low SNR and short block lengths, it has been observed (Ref. 9) that the computation distribution appears as if the code tail added to the effective signal but not so much to the noise. This would occur if the effective length of the decoding searches approached the block length. If we assume that the entire tail contributes to this effect, then there is an effective increase of over 0.5 dB for the 192-bit Pioneer 10 frame with a 24-bit tail, and about 0.1 dB for the 1152-bit Helios frame with a 32-bit tail.

The solid lines of Fig. 4 result from assuming  $E_b/N_0 = -2.5$  dB for the hypothetical data, equivalent to assuming a Pioneer block with the tail fully contributing to the effective SNR. The approximation parameters appear in Table 2. Figure 5 shows the computed distribution of the computation for a fixed symbol-error probability of 6% using the approximation of Eq. (3) (Table 2). The curves of Fig. 5 are now encouragingly close to the experimental curves of Fig. 3. To improve our confidence in performance curves computed in this fashion, we must determine experimentally the computation distribution at very low SNR. This is not an easily accomplished task because the number of decoder computations needed to acquire any fixed amount of computation distribution data is large for very low SNR.

At the moment, the best estimate of sequential decoding with noisy reference performance is obtained by computing from the approximation of Eq. (3) (Table 2). The distribution of computations for fixed bit SNR using this approximation is shown in Fig. 6a–c. The hypothetical augmented data set of Fig. 4 has also been approximated assuming  $E_b/N_0 = -2.0$  dB, for correspondence with long blocks, and the resultant parameters appear in Table 3. The computed distribution of the computation for this approximation is shown in Fig. 7a–c.

#### IV. Summary and Extensions

This article has presented a technique for analyzing the effect of a noisy carrier phase reference on sequential decoding. The technique produces a good approximation to frame erasure probability—when that probability is low—and a bound to the degradation caused by the noisy carrier reference for other regions of the distribution of computation curves. There is no previously published theory detailing the behavior of the distribution of computation in a noisy reference environment. However, the increase in SNR needed to counteract noisy reference losses and achieve a Pareto exponent  $\alpha = 1$  was bounded by Heller (Ref. 11). This bound is somewhat more pessimistic than that of Fig. 2 for  $\alpha \approx 1$ .

One of the inputs to this analysis is the experimental distribution of decoder computations per frame, which is necessarily a function of frame length. In order to develop an analysis of the effect of phase jitter on sequential decoding which is accurate for all ranges of the computation variable and which can be adapted to all

frame lengths, an accurate model of the sequential decoding process is needed. From observation of experimental distributions of sequential decoding computation, both on a per-bit and per-frame basis, I believe that the computations needed to sequentially decode a frame of data can be represented as the sum of a number of independent searches of varying length. If within a particular frame, a search of length  $l$  ends at depth  $M$ , then the number of computations needed to penetrate from depth  $M - j$  to  $M - j + 1$  is strongly dependent upon the numbers of computations needed to penetrate from depth  $M - 1$  to  $M$  whenever  $1 < j < l$ . Searches are characterized by two numbers, their length  $l$ , and the numbers of computations required to complete the search. Except for boundary effects, a frame contains some number of searches whose lengths total to the frame length. This hypothetical model can, and should, be tested by properly instrumenting a sequential decoder to determine the joint distribution of search length and number of computations. Assuming that this model is valid, the noisy reference problem could be easily attacked, since the search length  $l$  is almost certain to be short enough for the phase error  $\phi$  to be constant throughout each search. Alternatively, the search length  $l$  is the “integration time” parameter needed to apply interpolation techniques to decoding behavior when the phase error  $\phi$  varies during searches.

The joint distribution of search length and computations would be modified to include the effects of carrier phase jitter, employing the same numerical techniques just used, to produce a distribution family—in terms of bit SNR and carrier loop SNR—which can subsequently be used to derive all parameters of interest.

## References

1. Jacobs, I. M., and Berlekamp, E. R., "A Lower Bound to the Distribution of Computation for Sequential Decoding," *IEEE Trans. Info. Theory*, IT-13, pp. 167-174.
2. Heller, J. A., "Description and Operation of a Sequential Decoder Simulation Program," in *Supporting Research and Advanced Development*, Space Programs Summary 37-58, Vol. III, p. 42, Jet Propulsion Laboratory, Pasadena, Calif., Aug. 31, 1969.
3. Lumb, D., and Hoffman, L., "Preliminary Results on Coding for Pioneer F," NASA Ames Memo, Nov. 4, 1969.
4. Viterbi, A. J., *Principles of Coherent Communication*, Chap. 4, McGraw-Hill Book Co., Inc., New York, 1966.
5. Lindsey, W. C., "Block Coding for Space Communications," *IEEE Trans. Commun. Technol.*, Vol. COM-17, No. 2, April 1969, pp. 217-225.
6. Heller, J. A., and Jacobs, I. M., "Viterbi Decoding for Satellite and Space Communications," *IEEE Trans. Commun. Technol.*, Vol. COM-19, No. 5, Oct. 1971, pp. 835-848.
7. Tausworthe, R. C., "Efficiency of Noisy Reference Detection," in *Supporting Research and Advanced Development*, Space Programs Summary 37-54, Vol. III, pp. 195-201, Jet Propulsion Laboratory, Pasadena, Calif., Dec. 31, 1968.
8. Wilcher, J. H., *DSN DDA Performance for Pioneer Telemetry*, IOM 3384-72-056, June 26, 1972. (JPL internal document)
9. Layland, J. W., "Performance of an Optimum Buffer Management Strategy for Sequential Decoding," in *The Deep Space Network Progress Report*, Vol. IX, pp. 88-96, Jet Propulsion Laboratory, Pasadena, Calif., 1972.
10. Jacobs, I. M., "Sequential Decoding for Efficient Communication from Deep Space," *IEEE Trans. Commun. Technol.*, Vol. COM-15, No. 4, Aug. 1967, pp. 492-501.
11. Heller, J. A., "Sequential Decoding with Decision Directed Phase Estimation," in *Supporting Research and Advanced Development*, Space Programs Summary 37-48, Vol. III, pp. 181-187, Jet Propulsion Laboratory, Pasadena, Calif., Dec. 31, 1967.

**Table 1.  $A_{n,r}$  for Pioneer data**

$n$ $r$	-1	0	1
0	-4.48	5.81	-0.329
1	4.91	-5.01	0.806
2	-1.25	0.995	-0.685

**Table 2.  $A_{n,r}$  for short-frame, low-SNR extrapolation**

$n$ $r$	-1	0	1
0	0.44	0.179	-0.773
1	-0.127	0.603	1.44
2	0.016	-0.401	-0.903

**Table 3.  $A_{n,r}$  for long-frame, low-SNR extrapolation**

$n$ $r$	-1	0	1
0	-0.88	0.374	0.617
1	1.20	0.673	-0.181
2	-0.302	-0.512	-0.431

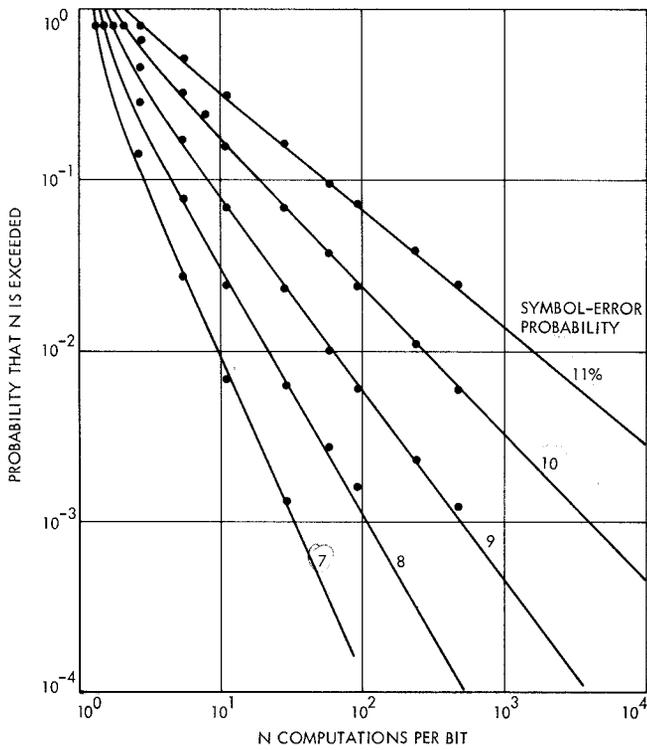


Fig. 1. Distribution of computations for sequential decoding of Pioneer 10 frame

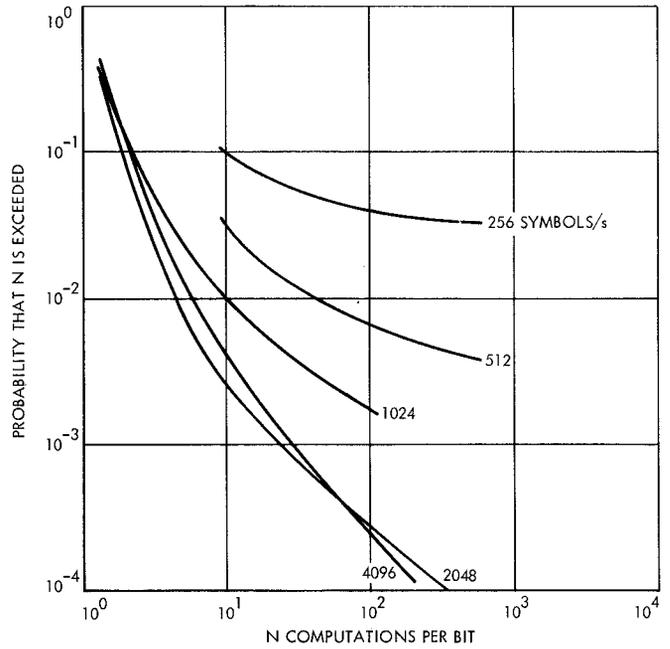


Fig. 3. Experimental computation distribution with noisy reference, at 6% symbol-error probability

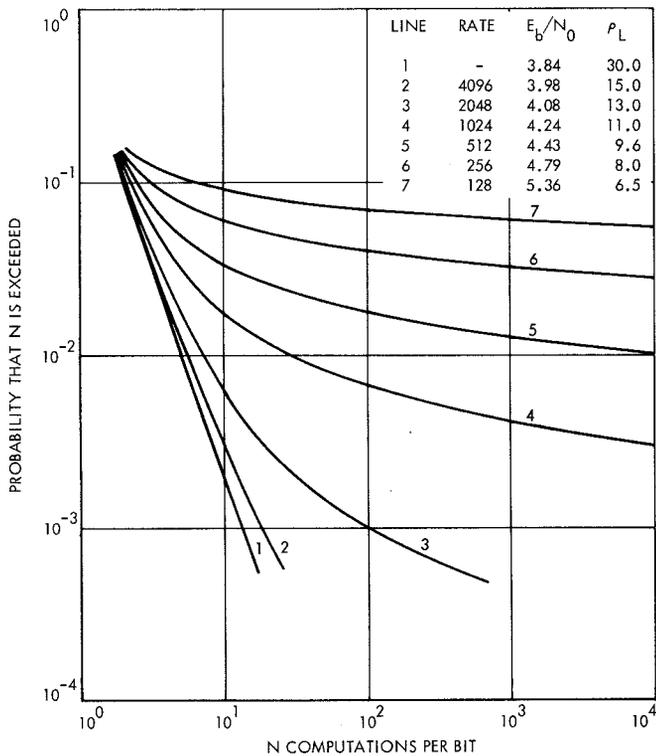


Fig. 2. Computed computation distribution with noisy reference, at 6% symbol-error probability

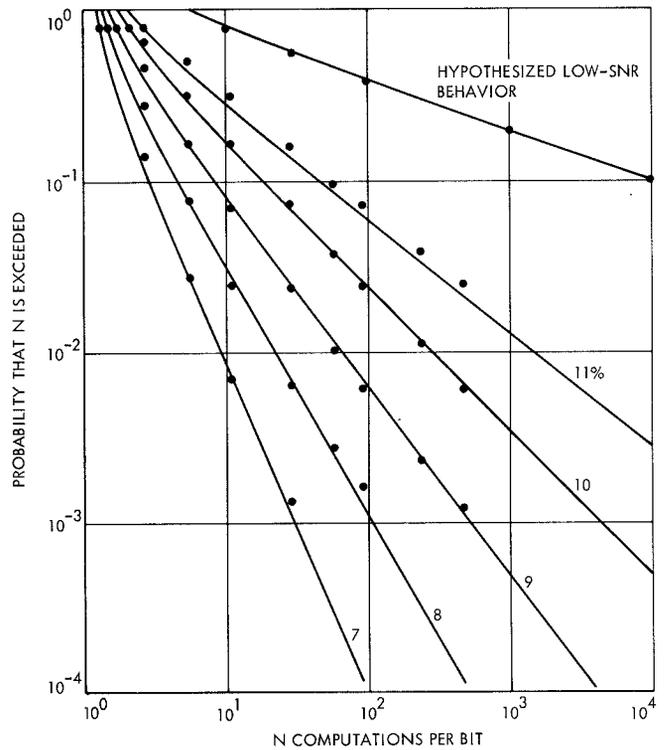


Fig. 4. Computation distribution for Pioneer 10 frame, with extrapolation to low SNR

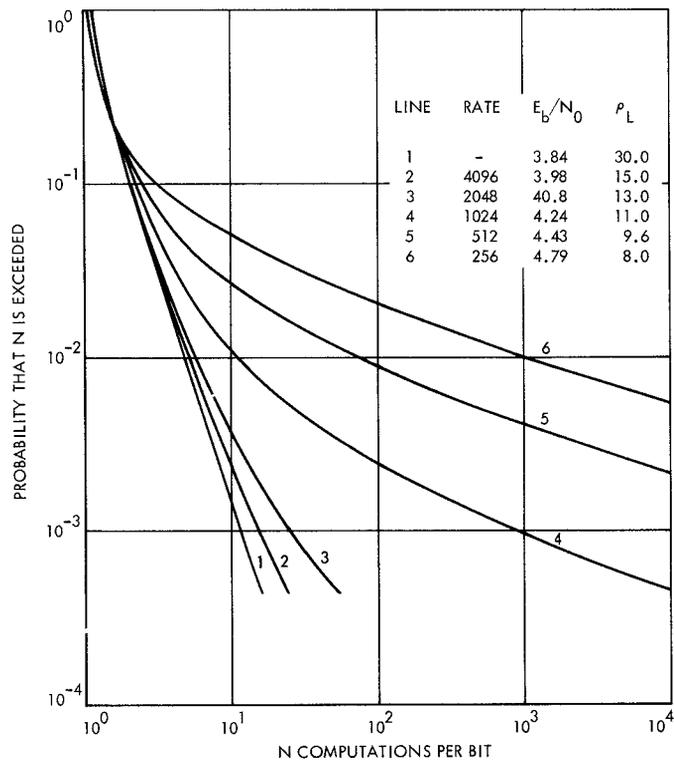


Fig. 5. Computed computation distribution with noisy reference using low-SNR extrapolation, at 6% symbol-error probability

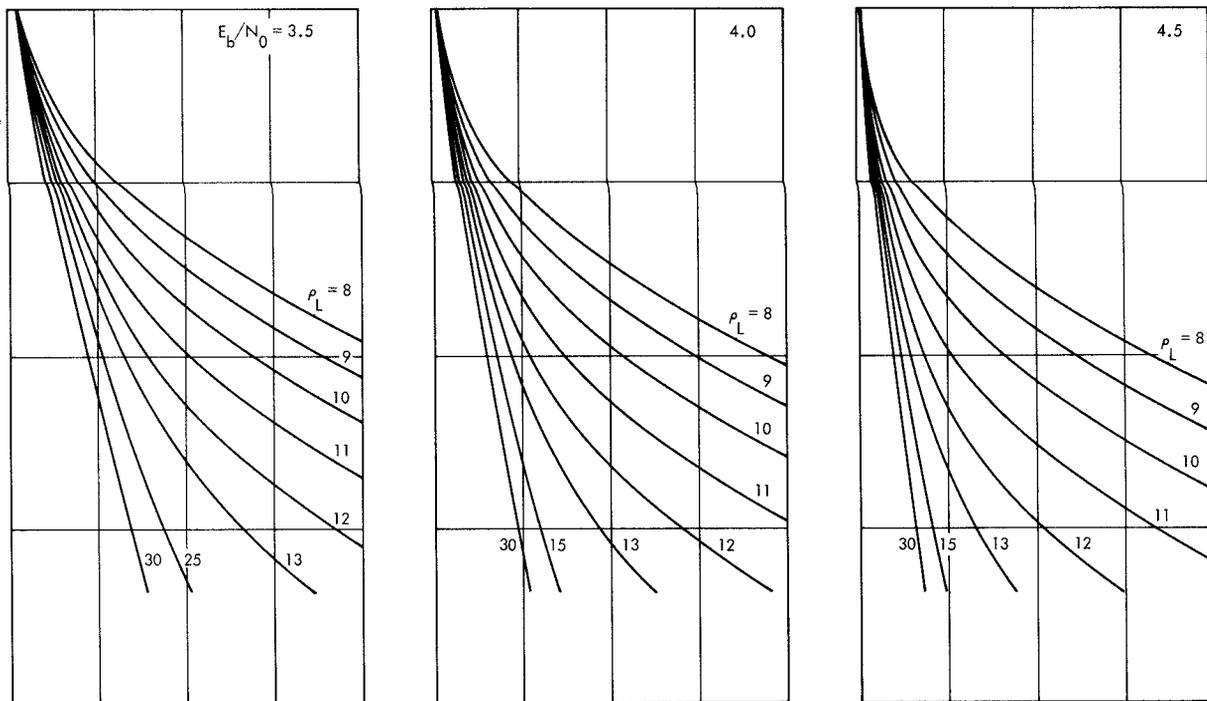


Fig. 6. Estimated computation distribution for short frames

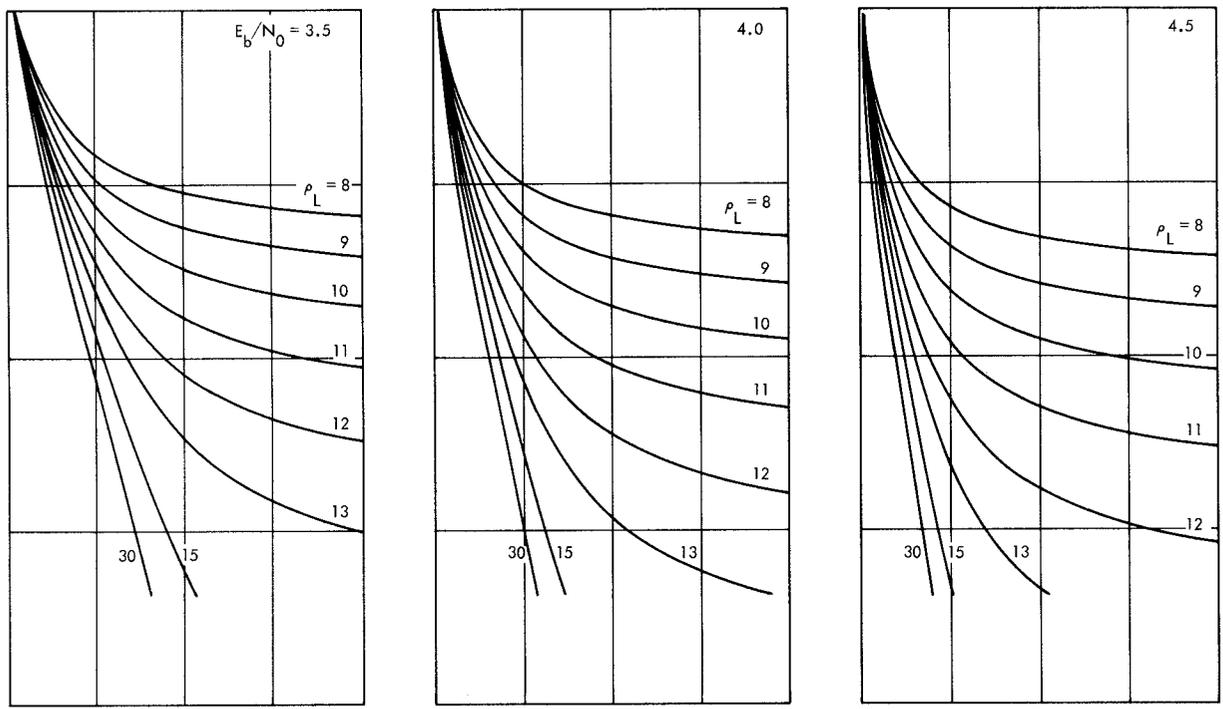


Fig. 7. Estimated computation distribution for long frames