High-Rate Convolutional Code Construction With the Minimum Required SNR Criterion

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New short constraint length, high-rate convolutional codes which minimize the required SNR are found and tabulated for rates 2/3, 3/4, and 4/5, and for constraint length K up to 10. When compared with previously reported codes, most of the new codes reduce the required SNR only slightly. However, there are some pairs of K and code rate for which the new codes require considerably less SNR. The most significant one is the new K = 8, rate 4/5 code which requires 1.25 dB less SNR than the known code with the same parameters, for a desired bit error rate of 10^{-6}.

I. Introduction

For a convolutional coding system employing a Viterbi decoder, the decoded bit error rate (BER) is well upper-bounded by the transfer function bound (Refs. 1; and 2, Chap. 4)

$$\mathrm{BER} \leqslant c_0 \cdot \frac{\partial}{\partial Z} T(D,Z) \Big|_{D=D_0, Z=1} = c_0 \cdot \sum_{i=d_f}^{\infty} a_i D_0^i \tag{1}$$

where the coefficient c_0 and transfer function $T(D,Z)$ depend on the code and type of channel used. The quantity D_0 is the Bhattacharyya bound (Ref. 2, p. 63) which depends on the channel only, d_f is the free distance of the code, and a_i is the number of bit errors in all incorrect coded symbol sequences with Hamming distance i. For an additive white Gaussian noise channel with binary PSK signaling (BPSK/AWGN channel) without quantization, we have (Refs. 1; and 2, p. 248)

$$D_0 = \exp(-E_s/N_0)$$

$$c_0 = Q(\sqrt{2d_f E_s/N_0}) \exp(d_f E_s/N_0)$$

where N_0 is the one-sided noise power spectral density, E_s is the received signal energy per channel symbol, and

$$Q(w) = \int_W^{\infty} \exp(-t^2/2)\, dt/\sqrt{2\pi}$$

Many researchers have used the maximum d_f criterion, or the criterion of maximum d_f together with minimizing the first few a_i's in Eq. (1) for determining the goodness of a code in their code search procedures. However, we have shown in Refs. 3 and 4 that, for low rate codes, these criteria do not necessarily lead to codes which minimize required signal-to-noise ratio (SNR) for a certain desired BER. Direct use of Eq. (1) for BER evaluation in the search procedure provides much better results.

This new minimum required SNR criterion is applied here to the searches for good high-rate codes, which are useful for systems with limited bandwidth. In the next sections, our notation is introduced and the code search procedure is briefly explained. Search results are then listed and discussed.

II. Preliminaries

Let m_0, k_0, and n_0 be the number of binary memory cells, inputs, and outputs of an $(m_0, k_0/n_0)$ convolutional encoder, where k_0/n_0 is the code rate r. A typical nonsystematic, time invariant encoder structure is shown in Fig. 1. A group of k_0 information bits is shifted into a shift register of length K (= $m_0 + k_0$), and outputs of n_0 modulo-2 adders are sampled and sequentially transmitted. The parameter K is called the constraint length of the code, while m_0 is called its memory length. Notice that the number of states in the Viterbi decoder trellis is 2^{m_0}. The low-rate codes considered in Refs. 3 and 4 are special cases with $k_0 = 1$.

Besides these key parameters, the code performance is determined by the connections from K shift registers to n_0 modulo-2 adders. These connections are often represented by an $n_0 \times K$ binary matrix **G**, called the code generator matrix, where "1" stands for connection and "0" for non-connection. As an illustration, a (2, 3/4) encoder and a (3, 2/3) encoder are shown in Fig. 2, whose code generator matrices are given, respectively, by

$$\mathbf{G} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

and

$$\mathbf{G} = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

For short hand notation, **G** is often represented by

$$(g(1), \ldots, g(j), \ldots, g(n_0))$$

where $g(j)$ is the jth row of **G**, in octal. For the codes in Fig. 2, **G** = (37,21,5,4) and **G** = (31,23,16), respectively. By "code search" we imply the search for a code generator **G** which provides good performance among the codes with same key parameters.

The transfer function bounding technique on the BER at the Viterbi decoder output will not be discussed here; it can be found in many references, including Refs. 5, 6, and 7.

III. Code Searching Procedure

In this study, we restricted our searches to high-rate codes with $n_0 = k_0 + 1$. Notice that the number of $(m_0, k_0/(k_0 + 1))$ codes in the whole code space is $2^{(m_0+k_0)\times(k_0+1)}$. For example, there are over 4 billion (5, 3/4) codes. Furthermore, according to our criterion, to test a code we have to evaluate the transfer function bound, which requires a matrix inversion. Therefore an exhaustive search is prohibitively difficult except for very small m_0 and k_0. Only partial searches are possible to obtain results in a reasonable length of time.

In the previous searches for low-rate codes, we developed several effective techniques for reducing the code search space. Many of these techniques are applied to the high-rate code searches with appropriate modifications.

First, for a given pair of m_0 and k_0, we made a list of some $r = 1$ codes (actually these are not codes since there is no redundancy) which are to be used for the generation of $r = k_0/(k_0 + 1)$ codes. In this list, by using the simple fact that exchanges of $g(j)$'s do not affect the performance of the code, identical codes are discarded. Also, codes with too small free distance (less than $d_x - 2$, where d_x is the maximum known free distance of $(m_0, k_0/(k_0 + 1))$ codes, or its bound if not known) are deleted. This procedure is based on the observation (Ref. 3) that the Hamming distance (from the all-zero output) on each branch (in the state diagram) of a lower rate code is always larger than or equal to that of the higher rate code, used as a seed for its generation. Catastrophic codes are not discarded at this time, as good $r = k_0/(k_0 + 1)$ codes are often found from catastrophic $r = k_0/k_0$ codes.

Each code in the list is used for generation of lower rate codes. Among the generated codes, identical codes and catastrophic codes are deleted. Codes with free distance smaller than $d_x - 1$ are also discarded. For each remaining code, the BER performance is found by the transfer function bound with a SNR at which the best code is expected to achieve BER of 10^{-6}.

IV. Search Results

The code search results are summarized in Table 1, where the code generators of best codes are shown with their free distance and the upper bound on the required bit SNR

(E_b/N_0, $E_b = E_s/r = E_s \cdot (k_0 + 1)/k_0$) value for desired BER of 10^{-6}. These new codes are compared with the codes reported in Refs. 8 and 9. If the best code in the sense of minimum required SNR does not have maximum free distance, then the best code among maximum free distance codes is also listed.

Notice that some of new codes have parameters never considered before. For all codes, we were able to find better codes than the previously reported codes. But the amount of SNR saving is usually very small except for a few cases. For the case of (4, 4/5) code (or equivalently $K = 8$, $r = 4/5$ code), we found a code which not only requires 1.25 dB less SNR but also has larger free distance than the previously reported code. Since we could not exhaust the code search space for most cases, there might be some better codes. However, we expect that better codes, if they exist, would improve the performance very slightly.

References

1. Viterbi, A. J., "Convolutional codes and their performance in communications systems," *IEEE Trans. Commun. Tech. COM-19*, pp. 751–772, Oct. 1971.

2. Viterbi, A. J., and Omura, J. K., *Principles of Digital Communication and Coding*, New York: McGraw-Hill Book Co., 1979.

3. Lee, P. J., "New short constraint, rate $1/N$ convolutional codes which minimize the required SNR for given desired bit error rate," *IEEE Trans. Commun. COM-33*, pp. 171–177, Feb. 1985.

4. Lee, P. J., "Further results on rate $1/N$ convolutional code construction with minimum required SNR criterion," *TDA Progress Report 42-80*, pp. 97–102, Jet Propulsion Laboratory, Pasadena, CA., Feb. 1985.

5. Omura, J. K., and Simon, M. K., Modulation/Demodulation Techniques for Satellite Communications, Part IV: Appendices, *JPL Publication 81–73*, Jet Propulsion Laboratory, Pasadena, CA., Nov. 1981.

6. Lee, P. J., "Transfer function bound for partial-unit-memory convolutional codes based on reduced state diagram," *TDA Progress Report 42-76*, pp. 70–76, Jet Propulsion Laboratory, Pasadena, CA., Feb. 1984.

7. Lee, P. J., "A very efficient transfer function bounding technique on bit error rate for Viterbi decoder, rate $1/N$ convolutional codes," *TDA Progress Report 42-79*, pp. 114–123, Jet Propulsion Laboratory, Pasadena, CA., Nov. 1984.

8. Paaske, E., "Short binary convolutional codes with maximal free distance for rate 2/3 and 3/4," *IEEE Trans. Info. Theory IT-20*, pp. 683–689, Sept. 1974.

9. Daut, D. G., Modestino, J. W., and Wismer, L. D., "New short constraint length convolutional code construction with selected rationale code rates," *IEEE Trans. Info. Theory IT-28*, pp. 794–800, Sept. 1982.

Table 1. Best (m_0, k_0/n_0) binary convolutional codes with $n_0 = k_0 + 1$ which minimize the required SNR for BER $\leq 10^{-6}$ and performance comparison to previously reported codes

$(m_0, k_0/n_0)$	d_f	E_b/N_0, dB	Note	Code generator G, in octal				
(1, 2/3)	2	9.023	A	5	3	2		
	2	9.032	D	6	3	2		
(2, 2/3)	3	7.360	A	15	13	12		
	3	7.570	D	17	15	6		
	3	7.798	P	16	13	11		
(3, 2/3)	4	6.292	A	31	23	16		
	4	6.320	D	33	22	15		
	4	6.341	P	37	22	11		
(4, 2/3)	5	5.870	A	61	46	37		
	5	5.888	P	61	56	27		
	5	6.169	D	75	72	27		
(5 2/3)	6	5.531	A	171	112	73		
	6	5.580	P	177	112	55		
(6, 2/3)	6	5.171	A	366	241	163		
	7	5.211	P	337	236	155		
(7, 2/3)	7	4.846	A	751	522	343		
	8	4.853	A	673	465	262		
	8	4.883	P	751	532	367		
(8, 2/3)	8	4.632	A	1671	1322	423		
(1, 3/4)	2	8.633	A	15	12	4	2	
	2	8.639	D	15	14	13	2	
(2, 3/4)	3	7.527	A	37	21	5	4	
	3	7.634	D	36	32	14	7	
(3, 3/4)	4	6.629	A	67	51	43	25	
	4	6.652	D	61	47	25	13	
(4, 3/4)	4	6.042	A	157	122	41	24	
	4	6.336	D	172	127	106	45	
(5, 3/4)	5	5.735	A	255	236	164	127	
	5	5.776	D	357	216	124	45	
	5	5.797	P	367	244	141	72	
(6, 3/4)	6	5.449	A	723	657	345	261	
	6	5.452	P	512	467	311	274	
(7, 3/4)	6	5.185	A	1752	1233	756	377	
(1, 4/5)	2	8.578	A	34	23	10	4	2
	2	8.820	D	36	26	13	11	4
(2, 4/5)	2	8.003	A	71	53	34	10	4
	2	8.507	D	67	57	52	26	15
(3, 4/5)	3	6.760	A	153	137	51	25	15
	3	6.838	D	174	132	56	23	13
(4, 4/5)	4	6.316	A	373	254	225	215	112
	3	7.561	D	337	274	255	237	156
(5, 4/5)	4	5.993	A	765	613	571	537	110
(6, 4/5)	5	5.710	A	1537	1351	1145	1053	730

NOTES: A Found by the author
P Found by Paaske (Ref. 8)
D Found by Daut, et al. (Ref. 9)

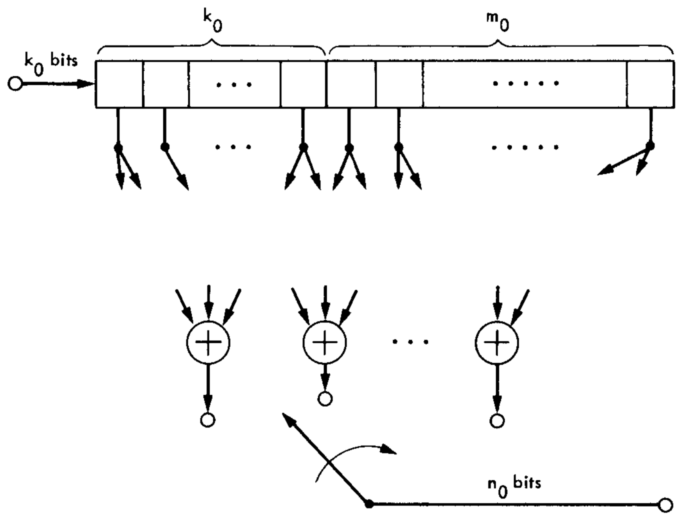

Fig. 1. A typical $(m_0, k_0/n_0)$ encoder structure

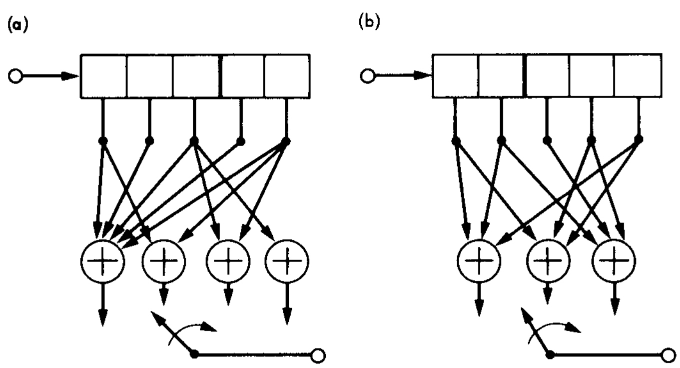

Fig. 2. Examples: (a) A (2, 3/4) code and (b) a (3, 2/3) code