Note on the Optimum Search Strategy for Uniformly Distributed CW Transmitters

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The relative probability of detecting randomly distributed CW transmitters as a function of the fraction of the sky which is searched (in a fixed time) is given. It is shown that the probability of detecting such a class of transmitters with a given receiving system is a maximum if the entire sky is searched. The particular case of a search in which the number of directions searched is equal to the telescope gain and the integration time per beam element is equal to the reciprocal of the channel bandwidth is discussed.

I. Introduction

It is important in designing a search program for SETI (Search for Extraterrestrial Intelligence) to understand the factors which effect the probability for detecting signals from distant transmitters. An optimum search strategy cannot be identified at the present time since the statistical properties of the transmitters and the motivations of the senders are not known. Nevertheless, a simple model, in which the transmitters are uniformly distributed in space and have a power law distribution of intrinsic transmitted power, appears to have sufficient generality to provide some guidance in designing a search program. We examine, in this note, the relative probability of detecting signals from such a class of transmitters in a fixed period of time, as a function of the fraction of sky which is searched. The scenario envisaged is that the observer may choose between scanning slowly (and achieving high sensitivity in a few directions) or scanning the entire sky (at reduced sensitivity).

We show that the probability of detecting such a class of CW transmitters is maximized if the entire sky is searched when the received signal-to-noise is low. This means that one is more likely to detect a weak CW signal by scanning the entire sky than by concentrating on a few areas. Since strong CW signals can be detected either by focusing the search in a few areas or by scanning the entire sky, the primary interest in this paper is on the weak signal case.

The detection of pulsed signals is not considered in this note. The detection of these signals may favor one scanning technique over another depending on factors such as the duty cycle of the transmitters.

II. The Probability Relationship

Derivation. Following Drake (Ref. 1), we assume that signals from civilizations radiating an equivalent isotropic radiated power (EIRP), P_o, are originating from uniformly spaced locations having a density of (n_o) transmitters per unit volume. We ignore the fact that the galaxy is a highly flattened disk and consider the spherical volume in the vicinity of the sun where stars are distributed more or less at random except for a tendency to cluster. This volume has a maximum radius of approximately 1 kiloparsec. The maximum range of a given search system is related to the minimum detectable flux, S_m, by the expression

$$R = \left(\frac{P_o}{4\pi S_m}\right)^{1/2} \tag{1}$$

and the total number of detectable signals within the spherical volume is given by

$$N_{\text{det}} = \left(4\pi \frac{n_o}{3}\right)\left(\frac{P_o}{4\pi S_m}\right)^{3/2} \tag{2}$$

The minimum detectable flux, S_m, depends on the diameter, D, of the receiving antenna, the system noise temperature, T_s, and the receiver channel bandwidth, B_s. Oliver and Billingham (Ref. 2) show that the minimum detectable (coherent) flux, defined as being equal to the rms fluctuations due to the noise, is equal to

$$S_m = \frac{k T_s B_s}{\frac{\pi D^2}{4}} \frac{1 + (1+n)^{1/2}}{n} \tag{3}$$

and the range at which a signal can be detected with a given receiving system is given by

$$R = \left(\frac{D}{4}\right)\left[\frac{P_o}{kT_s B_s} \frac{n}{(1 + (1+n)^{1/2})}\right]^{1/2} \tag{4}$$

In these expressions

k = Boltzmann's constant

T_s = system noise temperature

D = receiving antenna diameter

B_s = receiver channel bandwidth

t = integration time

$n = (B_s t)$ = number of independent samples averaged

Drake (Ref. 1) shows that the probability of success of an observation program is proportional to (a) the total frequency searched, (b) the total solid angle searched, and (c) the spherical volume defined by the radius R. These proportionalities lead to the following expression for the probability of success, P_s of a search over M different directions using a system having C channels (total bandwidth = CB_s), and an antenna beam solid angle equal to Ω_s:

$$P_s = K n_o M C \left(\frac{B_s}{B_t}\right)\left(\frac{\Omega_s}{\Omega_t}\right) R^3 \tag{5}$$

In this expression, K is a constant of proportionality, Ω_t (= 4π) is the total solid angle of the sky, and B_t is the total bandwidth range in which the signal is confined. The ratio $(M\Omega_s/\Omega_t)$ is the fraction of the sky which is surveyed and the ratio (CB_s/B_t) is the fraction of the total bandwidth which is surveyed.

Noting that the telescope gain, solid angle, the diameter are related through the equations

$$G = \left(\frac{\pi D}{\lambda}\right)^2 = \frac{1}{\Omega_s} \tag{6}$$

and the total observation time, ϕ, number of sky elements examined, M, and the number of independent samples at each sky element, n, are related through the equations

$$n = B_s t = B_s \frac{\phi}{M} \tag{7}$$

the probability of success can be written as follows

$$P_s = Kn_o MC \left(\frac{\Omega_s}{\Omega_t}\right)\left(\frac{B_s}{B_t}\right)\left(\frac{D^3}{64}\right)\left[\frac{P_o}{kT_s B_s} \frac{n}{1 + \sqrt{1+n}}\right]^{3/2} \tag{8}$$

$$= Kn_o MC \frac{\lambda^3}{(4\pi)^4}\left(\frac{B_s}{B_t}\right) G_2^{\,1/2} \times \left[\frac{P_o}{kT_s B_s} \frac{B_s \phi}{M + \sqrt{M^2 + MB_s\phi}}\right]^{3/2} \tag{9}$$

In the derivation of Eq. (9), we assumed that each transmitter had the same intrinsic power. It is easy to generalize the equation for the case of a continuous distribution of powers by rewritting Eq. (9) as a differential probability of success.

$$dP_s = KMC \frac{\lambda^3}{(4\pi)^4}\left(\frac{B_s}{B_t}\right) \times G^{1/2}\left[\frac{1}{kT_s B_s} \frac{B_s\phi}{M + \sqrt{M^2 + MB_s\phi}}\right]^{3/2} P^{3/2}\, P(P) dP \tag{10}$$

If the distribution of powers is a power law of the form

$$P(P) = K_1 P^{-\alpha} \qquad (11)$$

Equation (10) may be integrated between the limits P_u and P_L to yield the following expression for the probability of success:

$$P_s = KK_1 MC \frac{\lambda^3}{(4\pi)^4}\left(\frac{B_s}{B_t}\right) \times G^{1/2}\left[\frac{1}{kT_s B_s}\frac{B_s\phi}{M+\sqrt{M^2+MB_s\phi}}\right]^{3/2} \times \frac{P_u^{\,5/2-\alpha} - P_L^{\,5/2-\alpha}}{5/2-\alpha} \qquad (12)$$

It is seen from this equation that the spectral index, α, determines whether the strong sources or the weak sources will dominate the probability of success. If the spectral index is less than 5/2, then the strong sources, although less numerous will dominate. If the spectral index is greater than 5/2, then the weaker and more numerous sources will dominate. If the spectral index is exactly 5/2, then all sources regardless of their power will contribute uniformly to the probability of success.

In Eqs. (9) and (12), we have implicitly assumed that the telescope gain and solid angle are constant over the receiving bandwidth of the system ($C\,B_s$). This is true in most high performance antenna receiving systems, but may break down if the receiving bandwidth is very large. In this case, an integration over the wavelength needs to be performed.

Interpretation. Figure 1 shows the dependence of the probability on the number of different beam areas examined, M, for a number of different telescopes whose gains vary from 10^2 to 10^8. The value of $B_s\,\phi$ used is 10^8. We note in this figure that for any given telescope, the probability increases as M increases up to the point where the telescope gain becomes equal to the number of directions examined. M cannot exceed the gain since the gain sets the limit on the number of independent areas in the sky which can be examined.

Another constraint on the number of different directions which can be examined is superimposed by the system channel bandwidth, B_s. Since independent measurements cannot be made more frequently than the reciprocal of the bandwidth, the product of the channel bandwidth and total observation time is an upper limit on the number of independent directions in the sky which can be examined in a specified time.

$$M \leqslant B_s\phi \qquad (13)$$

This limit is in addition to the constraint set by the number of resolvable directions in the sky. If an attempt is made to increase M beyond this critical product, beam smearing will occur, and it will not be possible to differentiate spatial directions on the scale of the antenna beam size.

Figure 1 shows that it is advantageous to carry out a search by searching the entire sky (M = Gain) with the highest gain (largest) antenna available. This result holds provided that the telescope gain is less than the product $B_s\phi$. Figure 2 shows how the probability of success varies with the telescope gain under the following assumptions:

$$G < B_s\phi \qquad M = G \text{ (entire sky is surveyed)}$$

$$G > B_s\phi \qquad M = B_s\phi\,(M\Omega_s < 4\pi) = 10^8$$

Figure 2 shows that the probability of success increases as $(\text{Gain})^{3/4}$ up to the critical point, $B_s\phi = G$; at larger values of gain, the probability increases as $(\text{Gain})^{1/2}$. This figure shows that it is always better to use a large antenna rather than a small antenna. Of course there are mechanical considerations which may favor the use of a smaller antenna; these are not discussed here.

It is of interest to make a comparison between two identical receiving systems which both operate for the same time but with different amounts of the sky being examined. The ratio of the probability of success of a search which examines M different directions, to that of a search which examines the entire sky ($M = G =$ gain) is given by

$$\frac{P_s(M)}{P_s(G)} = \frac{M}{G}\left[\frac{G+\sqrt{G^2+GB_s\phi}}{M+\sqrt{M^2+MB_s\phi}}\right]^{3/2} \qquad (14)$$

Figure 3 shows this relationship for the particular case of a 60 dB gain antenna and a channel bandwidth-total observation time product, $B_s\phi = 10^8$. The figure clearly illustrates that the relative probability for detecting a weak signal increases as more directions are searched.

III. The Special Case of the Matched System

It is of interest to consider the special case in which the integration time per beam element is matched to the channel bandwidth, B_s, such that $B_s t = 1$, and the telescope gain is matched to the total observation time such that $G = M = B_s\phi$.

These two conditions define a matched system and lead to the following expression for the probability:

$$P = \frac{K^3}{(4\pi)^4} n_o C \left(\frac{B_s}{B_t}\right) \phi^{3/4} \left(\frac{P_o}{kT_s}\right)^{3/2} \tag{15}$$

It is noticed that this expression is independent of the telescope gain. This at first sight appears to be in contradiction with our previous result which stated that larger antennas produce a higher probability of success than smaller antennas. However, since the matched condition requires that channel bandwidth vary with the telescope gain, the previous result which assumed the bandwidth to be constant does not apply.

In order for matched systems to have the same probability of success, they must cover the same total bandwidth. Since matched low gain antenna systems utilize smaller channel bandwidths than higher gain antenna systems (channel bandwidth is proportional to Gain in matched systems), they must compensate by using a larger number of channels. The table below shows parameters for three matched systems with the same probability of success. The total observation time is assumed to be 10^7 seconds. The highest gain system is assumed to have a channel bandwidth of 1 Hz and a total bandwidth of 1 KHz.

Table 1 illustrates the ultranarrow channel bandwidths and large number of total channels which are required for a small antenna to have the same probability of success as a large system. Since the interstellar medium imposes a minimum usable bandwidth (i.e., Ref. 3), the bandwidth cannot be reduced indefinitely. Hence, in practice, a small antenna system cannot be made as sensitive as a large antenna system, even if the number of channels required imposes no limits.

IV. Discussion

We have shown in this note that the probability of detecting a weak signal from randomly distributed transmitters increases faster by increasing the fraction of sky that is searched than by increasing the sensitivity in a given direction. The reason why it is more advantageous to search a larger solid angle at reduced sensitivity than to search a smaller solid angle at increased sensitivity can be understood through the following argument. The number of detectable transmitters increases as the minimum detectable flux raised to the $-3/2$ power [Eqs. (1) and (2)]. Since the minimum detectable flux decreases as the inverse square root of time for weak signals, the number of detectable transmitters in a given direction increases as the observation time raised to the 3/4 power. On the other hand, the number of detectable transmitters is also proportional to the solid angle searched. Since the solid angle searched can be made to increase in proportion to the time by searching in different directions, the number of detectable transmitters can be made to increase in direct proportion to the time (rather than to the 3/4 power). Hence, the probability of success increases faster by scanning the entire sky than by concentrating in the search to a few directions. The ratio of the number of transmitters which could be detected with a given telescope system in a given time if the entire sky is searched to those that could be detected in the same time if a single direction is observed is proportional to the observation time raised to the 3/4 power.

Although this result was derived for the case of coherent radiation, it applies more generally to detection situations in which the minimum detectable flux decreases more slowly than the observation time raised to the $-2/3$ power. For example, the minimum detectable flux varies as the inverse square root of time for incoherent radiation. Hence the result holds for incoherent as well as (weak) coherent radiation. The result breaks down whenever the minimum detectable flux decreases faster than the observation time raised to the $-2/3$ power. One example where this occurs is in the detection of strong coherent signals. In this case, the flux varies inversely with time.

Aside from mechanical considerations, a large telescope can always be used to produce a higher probability of success than a smaller telescope. Also, it is possible that a small telescope which surveys the entire sky will have a higher probability of success than a larger telescope that concentrates a search in a few directions.

The probability of success of a "matched" system is the same for all telescopes regardless of their gain provided that they cover the same bandwidth. This requires that small telescopes use narrow bandwidths and thus a larger number of channels.

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References

1. Drake, F. D. (1983), Estimates of the Relative Probability of Success of the SETI Search Program, *SETI Science Working Group Report*, NASA Technical Report, Eds., F. Drake, J. H. Wolfe, C. L. Seeger, October 1983, p. 67.
2. Oliver, B. M., and Billingham, J. *Project Cyclops: A Design Study of a System for Detecting Extraterrestrial Intelligent Life*, prepared under Stanford/NASA/Ames Research Center, 1971 Summer Faculty Fellowship Program in Engineering System Design, NASA Report CR 114445, p. 56.
3. Drake, F. D., and Helou, G. (1977), The Optimum Frequencies for Interstellar Communications as Influenced by Minimum Bandwidths, Report 76, National Astronomy and Ionosphere Center, Cornell University, Ithaca, New York.

Table 1. Channel bandwidths and total of channels required for a small antenna to have the same probability of success as a large system

Gain	Observation Time per Sky Element	Number of Elements	Bandwidth	Number of Channels
10^6	10	10^6	1	10^3
10^5	100	10^5	0.1	10^4
1	10^7	1	10^{-6}	10^9

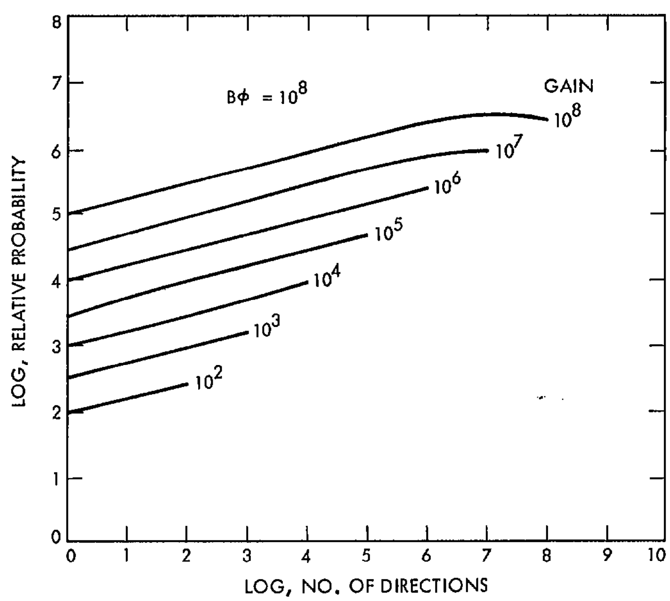

Fig. 1. Relative probability of success for a number of different antennas whose gains vary from 10^2 to 10^8 as a function of the number of directions in the sky which are searched. The search time–channel bandwidth product is taken to 10^8.

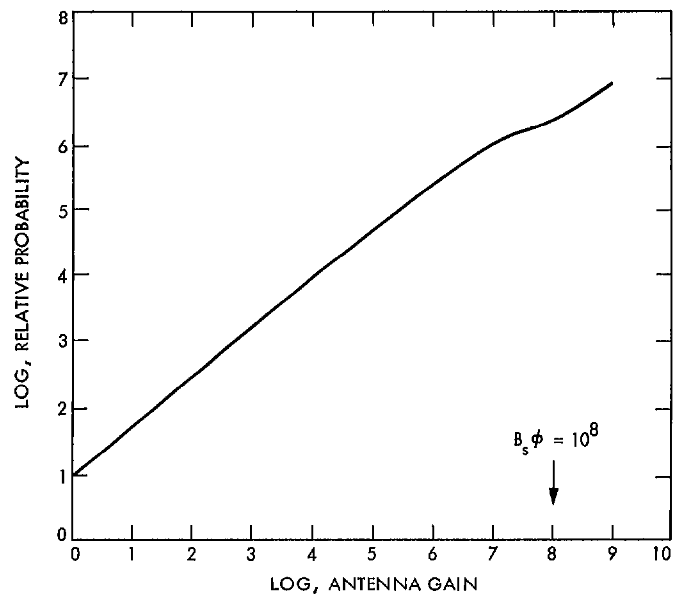

Fig. 2. Relative probability of success as a function of the antenna gain. The search time–channel bandwidth product is taken to be 10^8.

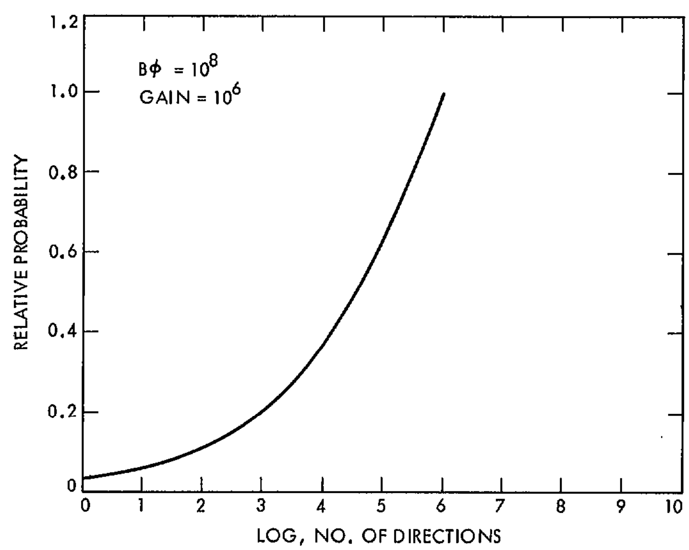

Fig. 3. Relative probability of success as a function of the number of directions in the sky which are searched for a 60 dB antenna. Curve is normalized to unit probability at 10^6 directions which corresponds to the entire sky being searched. The search time–channel bandwidth product is taken to be 10^8.